

# Relativistische Umlaufbahnen und Schwarze Löcher

Die aufregende Geschichte eines Partikels (Photon), welches mit dem Schwarzen Loch Samba tanzt

Dr. Wilfried Tenten Juni 2005 bis Januar 2007

Aufbauend auf der Lagrange DGL und deren numerischer Lösung werden Trajektorien von Körpern sowohl mit der klassischen Newton Theorie als auch mit der ART inklusive Schwarzschild und Kerr Metrik gerechnet.

Die Lagrange DGL "L" der Kerr'schen Lösung der ART beschreibt ein spärisches Objekt der Masse M mit einem Drehwinkel-Moment (Spin Moment) J:

$$L = -\frac{\left(1 - \frac{2M}{r}\right)\left(\frac{d}{d\tau} t(\tau)\right)^2}{2} - \frac{2aM\left(\frac{d}{d\tau} t(\tau)\right)\left(\frac{d}{d\tau} \phi(\tau)\right)}{r} + \frac{r^2\left(\frac{d}{d\tau} r(\tau)\right)^2}{2\Delta} + \frac{\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\left(\frac{d}{d\tau} \phi(\tau)\right)^2}{2}$$

mit:

$$a = \frac{J}{M} \quad \text{and} \quad \Delta = r^2 - 2Mr + a^2$$

.  
Ist  $a = 0$ . heißt das, dieses Objekt rotiert nicht. Die Kerr Lösung mutiert zur Schwarzschild Lösung. Für  $c = \text{unendlich}$  (infinity) und  $t = \text{tau}$  reduziert sich die Schwarzschildlösung auf die Newton Lösung der klassischen Mechanik. Das heißt als Differenz zwischen der kinetischen (Bewegungs-) und potentiellen (Lage oder Ruhe) energie (das ist die Aussage der Lagrange Gleichung). Die Gleichung der Bewegung erfolgt aus einer Vereinfachung der euler-Lagrange Beziehung:

$$\frac{d}{dt} \frac{\partial}{\partial q'_i} L - \frac{\partial}{\partial q_i} L = 0$$

wobei  $q$  eine Koordinate darstellt und  $q'$  deren Differential zur Zeit ist.

```
> restart: with(plots): with(plottools):  
Warning, the name changecoords has been redefined  
  
Warning, the name arrow has been redefined
```

Zunächst Definitionen: kinetische und potentielle Energie sowie Lagrange Gleichung

```
> T := 1/2*(diff(r(t),t)^2 + r(t)^2*diff(phi(t),t)^2);
```

$$T := \frac{1}{2} \left( \frac{d}{dt} r(t) \right)^2 + \frac{1}{2} r(t)^2 \left( \frac{d}{dt} \phi(t) \right)^2$$

```
> V := -G*M/r(t);
```

$$V := -\frac{GM}{r(t)}$$

```
> L := T - V;
```

$$L := \frac{1}{2} \left( \frac{d}{dt} r(t) \right)^2 + \frac{1}{2} r(t)^2 \left( \frac{d}{dt} \phi(t) \right)^2 + \frac{GM}{r(t)}$$

```
> L1 := subs({r(t)=var1, diff(r(t),t)=var2, phi(t)=var3, diff(phi(t),t)=var4}, L);
```

$$L1 := \frac{\text{var2}^2}{2} + \frac{\text{var1}^2 \text{var4}^2}{2} + \frac{GM}{\text{var1}}$$

Diese Substitution umfasst den radius r(t), das Potential phi(t). Dabei fällt wegen diff(phi(t),t) die Variable 3 weg.

Diese Euler Lagrange Beziehung wird für die Newton-Lagrange Gleichungen vorbereitet.

Dazu wird die Lagrange Funktion

partiell zuerst nach var4 differenziert und danach rücks substituiert.

```
> e11 := diff(L1, var4);
```

```
> e12 := diff(L1, var3);
```

```
> e13 :=
```

```
subs({var1=r(t),var2=diff(r(t),t),var3=phi(t),var4=diff(phi(t),t)},e11);
```

```
> eq14 := e13 = l;
```

```
> e21 := diff(L1, var2):
```

```
> e22 := diff(L1, var1):
```

```
> e23 := subs({var1=r(t), var2=diff(r(t),t), var3=phi(t), var4=diff(phi(t),t)}, e21):
```

```
> e24 := subs({var1=r(t), var2=diff(r(t),t), var3=phi(t), var4=diff(phi(t),t)}, e22):
```

```
> e25 := diff(e23,t):
```

```
> eq26 := e25 - e24 = 0;
```

$$e11 := \text{var1}^2 \text{var4}$$

$$e12 := 0$$

$$e13 := r(t)^2 \left( \frac{d}{dt} \phi(t) \right)$$

$$eq14 := r(t)^2 \left( \frac{d}{dt} \phi(t) \right) = l$$

$$eq26 := \left( \frac{d^2}{dt^2} r(t) \right) - r(t) \left( \frac{d}{dt} \phi(t) \right)^2 + \frac{GM}{r(t)^2} = 0$$

eq31 und eq32 sind entkoppelte DGLs aus eq14

```
> eq31 := isolate(eq14, diff(phi(t),t));
```

$$eq31 := \frac{d}{dt} \phi(t) = \frac{l}{r(t)^2}$$

```
> eq32 := eval(eq26, eq31);
```

$$eq32 := \left( \frac{d^2}{dt^2} r(t) \right) - \frac{l^2}{r(t)^3} + \frac{GM}{r(t)^2} = 0$$

Die init Werte werden behalten, die DGLs werden numerisch gelöst und es wird ein schön Bildchen gemalt. G und M werden normiert zu 1

```
> G:=1; M:=1;
```

```
G := 1
```

```
M := 1
```

```
> eq41 := r(0) = 26;
```

```
> eq42 := D(r)(0) = 0;
```

```
> eq43 := phi(0) = 0;
```

```
> eq44 := D(phi)(0) = 0.0071;
```

```
> en := eval(T + V, {r(t)=rhs(eq41), diff(r(t),t)=rhs(eq42),
diff(phi(t),t)=rhs(eq44)});
```

```
> l := eval(lhs(eq14), {r(t)=rhs(eq41),
diff(phi(t),t)=rhs(eq44)});
```

```
eq41 := r(0) = 26
```

```
eq42 := D(r)(0) = 0
```

```
eq43 := phi(0) = 0
```

```
eq44 := D(phi)(0) = 0.0071
```

```
en := -0.02142295846
```

```
l := 4.7996
```

Die Exzentrizität wird nicht benötigt, wird aber als Referenz aufgeführt

```
> epsilon := sqrt(1 + 2*en*l^2/((G*M)^2));
```

```
epsilon := 0.1139938402
```

```
> ini1 := eq41, eq42, eq43;
```

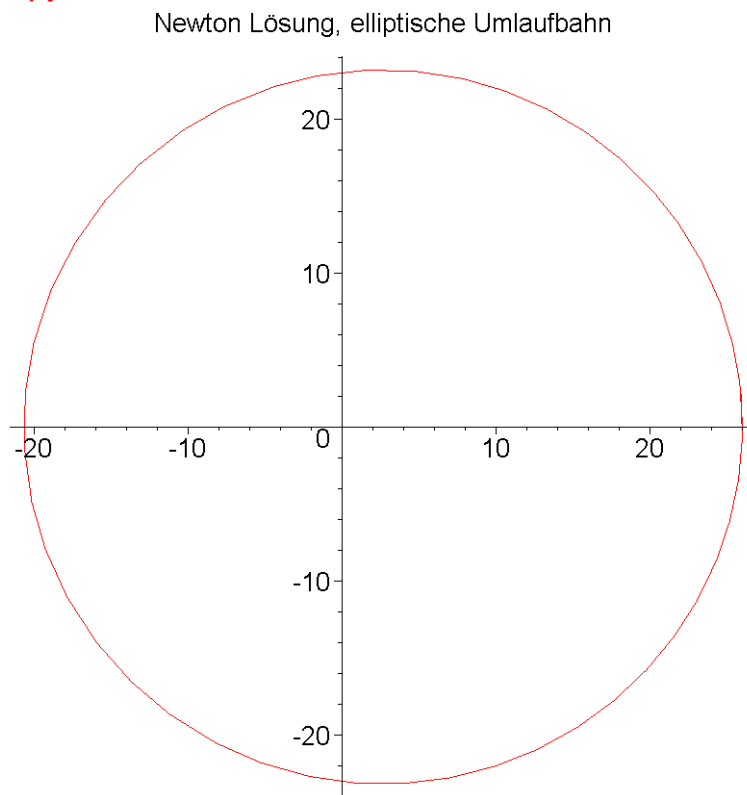
```
ini1 := r(0) = 26, D(r)(0) = 0, phi(0) = 0
```

```
> eq51:=dsolve({eq31, eq32, ini1}, {r(t), phi(t)}, numeric,
output=listprocedure);
```

```
eq51 := [ t = (proc(t) ... end proc), phi(t) = (proc(t) ... end proc),
```

```
r(t) = (proc(t) ... end proc),  $\frac{d}{dt} r(t) = (\text{proc}(t) \dots \text{end proc})$  ]
```

```
> polarplot([rhs(eq51(t))[3]], rhs(eq51(t))[2]), t=0..720],
scaling=constrained, title="Newton Lösung, elliptische
Umlaufbahn");
```



```
> pnewton := polarplot([rhs(eq51(t))[3]], rhs(eq51(t))[2]),
t=0..720], scaling=constrained, color=black, legend="Newton
Lösung", linestyle=3, thickness=2):
```

## [- Schwarzschild Metrik

```
> G := 'G': M := 'M':
```

Die Berechnung der Schwarzschild Metrik ist nahezu identisch mit der Newton Mechanik  
Wir nutzen die Lagrange Beziehung:

```
> f := 1/2*(-(c^2 - 2*G*M/r(tau))*diff(t(tau),tau)^2
+ 1/(1 - 2*G*M/(c^2*r(tau)))*diff(r(tau),tau)^2 +
r(tau)^2*diff(phi(tau),tau)^2);
```

$$f := -\frac{1}{2} \left( c^2 - \frac{2GM}{r(\tau)} \right) \left( \frac{d}{d\tau} t(\tau) \right)^2 + \frac{1}{2} \frac{\left( \frac{d}{d\tau} r(\tau) \right)^2}{1 - \frac{2GM}{c^2 r(\tau)}} + \frac{1}{2} r(\tau)^2 \left( \frac{d}{d\tau} \phi(\tau) \right)^2$$

```
> f1 := subs({t(tau)=var1, diff(t(tau),tau)=var2, r(tau)=var3,
diff(r(tau),tau)=var4,
phi(tau)=var5, diff(phi(tau),tau)=var6}, f):
```

Die Euler-Lagrange Beziehungen werden gerechnet (e11 bis eq34)

```

> e11 := diff(f1, var2):
> e12 := diff(f1, var1):
> e13 := subs({var1=t(tau), var2=diff(t(tau),tau), var3=r(tau),
var5=phi(tau),
var4=diff(r(tau),tau),
var6=diff(phi(tau),tau)}, e11):
> eq14 := e13 = -a;
>
> e21 := diff(f1, var4):
> e22 := diff(f1, var3):
> e23 := subs({var1=t(tau), var2=diff(t(tau),tau), var3=r(tau),
var5=phi(tau),
var4=diff(r(tau),tau),
var6=diff(phi(tau),tau)}, e21):
> e24 := subs({var1=t(tau), var2=diff(t(tau),tau), var3=r(tau),
var5=phi(tau),
var4=diff(r(tau),tau),
var6=diff(phi(tau),tau)}, e22):
> e25 := diff(e23, tau):
> eq26 := e25 - e24 = 0;
>
> e31 := diff(f1, var6):
> e32 := diff(f1, var5):
> e33 := subs({var1=t(tau), var2=diff(t(tau),tau), var3=r(tau),
var5=phi(tau),
var4=diff(r(tau),tau),
var6=diff(phi(tau),tau)}, e31):
> eq34 := e33 = h;

```

$$eq14 := -\left(c^2 - \frac{2GM}{r(\tau)}\right) \left(\frac{d}{d\tau} t(\tau)\right) = -a$$

$$eq26 := -\frac{\left(\frac{d}{d\tau} r(\tau)\right)^2 GM}{\left(1 - \frac{2GM}{c^2 r(\tau)}\right)^2 c^2 r(\tau)^2} + \frac{\frac{d^2}{d\tau^2} r(\tau)}{1 - \frac{2GM}{c^2 r(\tau)}} + \frac{GM \left(\frac{d}{d\tau} t(\tau)\right)^2}{r(\tau)^2} - r(\tau) \left(\frac{d}{d\tau} \phi(\tau)\right)^2 = 0$$

$$eq34 := r(\tau)^2 \left(\frac{d}{d\tau} \phi(\tau)\right) = h$$

Dieses DGL System wird in die folgenden 3 DGLs aufgedröselt

```

> eq53 := isolate(eq14, diff(t(tau),tau));

```

$$eq53 := \frac{d}{d\tau} t(\tau) = - \frac{a}{-c^2 + \frac{2GM}{r(\tau)}}$$

```
> eq54 := isolate(eq34, diff(phi(tau),tau));
```

$$eq54 := \frac{d}{d\tau} \phi(\tau) = \frac{h}{r(\tau)^2}$$

```
> eq55 := subs({eq53, eq54}, eq26);
```

$$eq55 := - \frac{\left(\frac{d}{d\tau} r(\tau)\right)^2 GM}{\left(1 - \frac{2GM}{c^2 r(\tau)}\right)^2 c^2 r(\tau)^2} + \frac{\frac{d^2}{d\tau^2} r(\tau)}{1 - \frac{2GM}{c^2 r(\tau)}} + \frac{GM a^2}{r(\tau)^2 \left(-c^2 + \frac{2GM}{r(\tau)}\right)^2} - \frac{h^2}{r(\tau)^3} = 0$$

Die Init Werte werden beibehalten, wir lösen wieder numerisch und bedienen uns der Normierung von  $M=1$   $G=1$  und  $c=1$ . Dann malen wir wieder ein schön Bildchen

```
> M := 1; G := 1; c := 1;
```

$$M := 1$$

$$G := 1$$

$$c := 1$$

```
> eq77 := r(0) = 26;
```

$$eq77 := r(0) = 26$$

```
> eq78 := D(r)(0) = 0;
```

$$eq78 := D(r)(0) = 0$$

```
> eq79 := phi(0) = 0;
```

$$eq79 := \phi(0) = 0$$

```
> eq80 := D(phi)(0) = 0.0071;
```

$$eq80 := D(\phi)(0) = 0.0071$$

```
> h := eval(lhs(eq34), {r(tau)=rhs(eq77),
diff(phi(tau),tau)=rhs(eq80)});
```

$$h := 4.7996$$

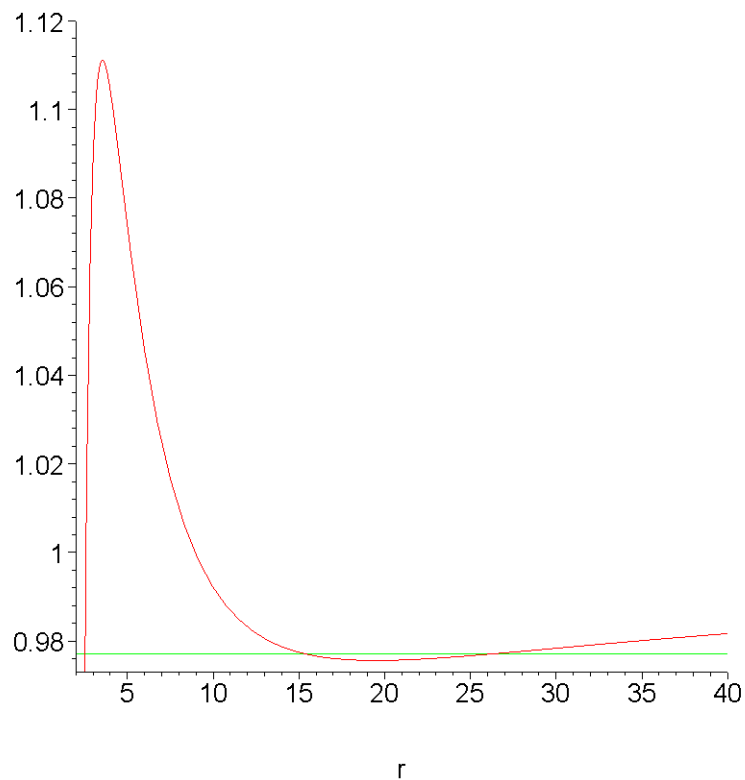
```
> a := sqrt((1 - 2*M/rhs(eq77))*(1 + h^2/rhs(eq77)^2));
```

$$a := 0.9770019258$$

es wird jetzt das effektive Potential mit seinem Umkehrpunkt visualisiert

```
> Vsqr := sqrt((1 - 2*M/r)*(1 + h^2/r^2));
```

```
> plot([Vsqr, a], r=2..40, 0.973..1.12, legend=["effektives
Potential", "Energie"]);
```



— effektives Potential  
— Energie

```

> fsolve(Vsq=a, r, 10..20);
                                15.46812439
> fsolve(Vsq=a, r, 20..30);
                                25.999999991
> ini := eq77, eq78, eq79;
                                ini := r(0) = 26, D(r)(0) = 0, phi(0) = 0
> eq91 := dsolve({eq54, eq55, ini}, {r(tau), phi(tau)},
numeric,
                                output=listprocedure);
eq91 := [ tau = (proc(tau) ... end proc), phi(tau) = (proc(tau) ... end proc),
          r(tau) = (proc(tau) ... end proc),  $\frac{d}{d\tau} r(\tau) = (\text{proc}(\tau) \dots \text{end proc}) ]$ 
> rhs(eq91[3])(0);
                                26.
>
> x1:=rhs(eq91[2])(200); x2:=rhs(eq91[3])(200);
> x3:=rhs(eq91[3])(1000);x4:=rhs(eq91[3])(2000);
  
```

```

x1 := 1.70330704882100381
x2 := 20.0929200675924129
x3 := 15.6676349325441998
x4 := 25.6917891767140958
> alpha:=45*Pi/180;

$$\alpha := \frac{\pi}{4}$$

> eq:=x=(rhs(eq91[3])(alpha));
x1:=solve(eq,x);evalf(cos(alpha));evalf(sin(alpha));
eq := x = 25.9999013472277270
x1 := 25.99990135
0.7071067810
0.7071067810
> x0:=evalf(1+2/3)*cos((rhs(eq91[2])(alpha)));
x0 := 1.666640754
> psch1 := polarplot([rhs(eq91(tau)[3]), rhs(eq91(tau)[2]),
tau=0..300],
scaling=constrained, axesfont=[TIMES, ROMAN, 12],
title="Umlaufbahnen um einen
Massekörper",legend="Schwarzschild"):
> psch2 := disk([15.47*1.56*cos(alpha), 15.47*1.56*sin(alpha)],
0.5, color=black):
> psch3 := disk([15.47*cos(x3), 15.47*sin(x3)], 0.5,
color=black):
> psch4 := disk([15.47*cos(x3), 15.47*sin(x3)], 0.5,
color=black):
> psch5 := disk([15.47*cos(x3), 15.47*sin(x3)], 0.5,
color=black):
> pns := disk([0,0],5.8, color=orange):

```

Nun die ganze Schönheit der Bahnkurvenschar um einen Massekörper. Das ist der rote (nicht der grüne!!) Punkt. Die Newton Lösung stellt die kreisförmige (schwarze) Bahn dar. Die rote Linie mit einigen exemplarisch gemalten Probekörpern die Bahn der Schwarzschildlösung. Deutlich sind die Unterschiede zu erkennen.

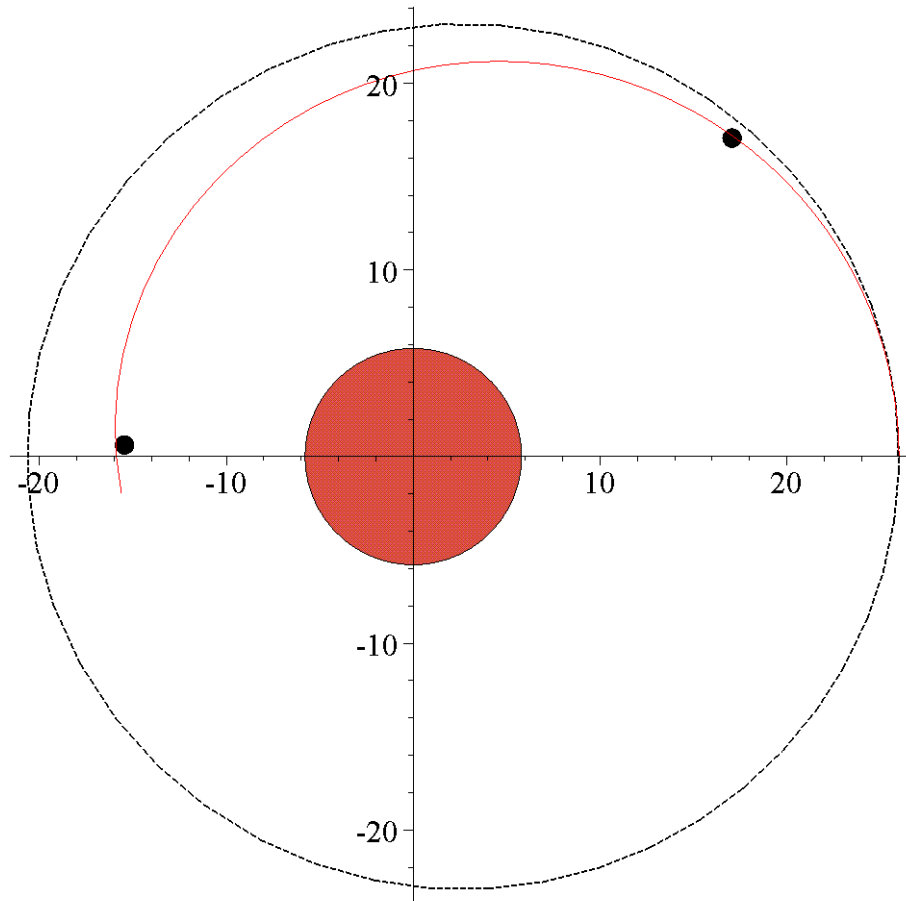
```

> display([psch1, psch2, psch3, psch4, psch5, pns, pnewton]);

```



## Umlaufbahnen um einen Massekörper



— Schwarzschild  
- - - Newton Lösung

```
> pschwarz := polarplot([rhs(eq91(tau)[3]), rhs(eq91(tau)[2]),  
tau=0..1000],  
scaling=constrained, color=blue,  
legend=Schwarzschild):  
> pschwarz3d := spacecurve([rhs(eq91(s)[3]), rhs(eq91(s)[2]),  
sqrt(8*(rhs(eq91(s)[3])-2)), s=0..2500],  
coords=cylindrical, color=black,  
numpoints=400):
```

## - Kerr Metrik

```
> M := 'M': l := 'l':
```

The same procedure as every year:

Das gleiche mit der Kerr Metrik. Das Einzige was wir ändern ist die Lagrange Beziehung

```
> a := J/M; Delta := r(tau)^2 - 2*M*r(tau) + a^2;
```

$$a := \frac{J}{M}$$

$$\Delta := r(\tau)^2 - 2 M r(\tau) + \frac{J^2}{M^2}$$

```
> Delta := r(tau)^2 - 2*M*r(tau) + J^2/M^2;
```

$$\Delta := r(\tau)^2 - 2 M r(\tau) + \frac{J^2}{M^2}$$

```
> f := 1/2*(-(1 - 2*M/r(tau))*diff(t(tau),tau)^2
+ r(tau)^2/(Delta)*diff(r(tau),tau)^2 + (r(tau)^2 + a^2 +
2*M*a^2/r(tau))*diff(phi(tau),tau)^2 - 4*a*M/r(tau)*diff(t(tau)
, tau)*diff(phi(tau), tau));
```

$$f := -\frac{1}{2} \left( 1 - \frac{2M}{r(\tau)} \right) \left( \frac{d}{d\tau} t(\tau) \right)^2 + \frac{1}{2} \frac{r(\tau)^2 \left( \frac{d}{d\tau} r(\tau) \right)^2}{r(\tau)^2 - 2 M r(\tau) + \frac{J^2}{M^2}} + \frac{1}{2} \left( r(\tau)^2 + \frac{J^2}{M^2} + \frac{2J^2}{M r(\tau)} \right) \left( \frac{d}{d\tau} \phi(\tau) \right)^2 - \frac{2J \left( \frac{d}{d\tau} t(\tau) \right) \left( \frac{d}{d\tau} \phi(\tau) \right)}{r(\tau)}$$

```
> f1 := subs({t(tau)=var1, diff(t(tau),tau)=var2, r(tau)=var3,
diff(r(tau),tau)=var4,
phi(tau)=var5, diff(phi(tau),tau)=var6}, f):
```

Entwicklung der Euler-Lagrange Gleichungen e11 bis e34

```
> e11 := diff(f1, var2):
> e12 := diff(f1, var1):
> e13 := subs({var1=t(tau), var2=diff(t(tau),tau), var3=r(tau),
var5=phi(tau),
var4=diff(r(tau),tau), var6=diff(phi(tau),tau)},
e11):
```

Da passiert immer die Änderung: erst war es 1, dann -a und jetzt -e

```
> eq14 := e13 = -e;
```

$$eq14 := -\left( 1 - \frac{2M}{r(\tau)} \right) \left( \frac{d}{d\tau} t(\tau) \right) - \frac{2J \left( \frac{d}{d\tau} \phi(\tau) \right)}{r(\tau)} = -e$$

```
> e21 := diff(f1, var4):
> e22 := diff(f1, var3):
> e23 := subs({var1=t(tau), var2=diff(t(tau),tau), var3=r(tau),
var5=phi(tau),
var4=diff(r(tau),tau), var6=diff(phi(tau),tau)},
```

e21):

```
> e24 := subs({var1=t(tau), var2=diff(t(tau),tau), var3=r(tau),
var5=phi(tau),
var4=diff(r(tau),tau), var6=diff(phi(tau),tau)},
```

e22):

```
> e25 := diff(e23, tau):
```

```
> eq26 := e25 - e24 = 0;
```

$$eq26 := \frac{r(\tau) \left( \frac{d}{d\tau} r(\tau) \right)^2}{r(\tau)^2 - 2 M r(\tau) + \frac{J^2}{M^2}} - \frac{r(\tau)^2 \left( \frac{d}{d\tau} r(\tau) \right) \left( 2 r(\tau) \left( \frac{d}{d\tau} r(\tau) \right) - 2 M \left( \frac{d}{d\tau} r(\tau) \right) \right)}{\left( r(\tau)^2 - 2 M r(\tau) + \frac{J^2}{M^2} \right)^2}$$

$$+ \frac{r(\tau)^2 \left( \frac{d^2}{d\tau^2} r(\tau) \right)}{r(\tau)^2 - 2 M r(\tau) + \frac{J^2}{M^2}} + \frac{M \left( \frac{d}{d\tau} t(\tau) \right)^2}{r(\tau)^2} + \frac{1}{2} \frac{r(\tau)^2 \left( \frac{d}{d\tau} r(\tau) \right)^2 (2 r(\tau) - 2 M)}{\left( r(\tau)^2 - 2 M r(\tau) + \frac{J^2}{M^2} \right)^2}$$

$$- \frac{1}{2} \left( 2 r(\tau) - \frac{2 J^2}{M r(\tau)^2} \right) \left( \frac{d}{d\tau} \phi(\tau) \right)^2 - \frac{2 J \left( \frac{d}{d\tau} t(\tau) \right) \left( \frac{d}{d\tau} \phi(\tau) \right)}{r(\tau)^2} = 0$$

```
> e31 := diff(f1, var6):
```

```
> e32 := diff(f1, var5):
```

```
> e33 := subs({var1=t(tau), var2=diff(t(tau),tau), var3=r(tau),
var5=phi(tau),
var4=diff(r(tau),tau), var6=diff(phi(tau),tau)},
```

e31):

```
> eq34 := e33 = 1;
```

$$eq34 := \left( r(\tau)^2 + \frac{J^2}{M^2} + \frac{2 J^2}{M r(\tau)} \right) \left( \frac{d}{d\tau} \phi(\tau) \right) - \frac{2 J \left( \frac{d}{d\tau} t(\tau) \right)}{r(\tau)} = 1$$

es erfolgt eine ganz ähnliche Art der Entkopplung: eq53 bis eq 55)

```
> eq53 := isolate(eq14, diff(t(tau),tau));
```

$$eq53 := \frac{d}{d\tau} t(\tau) = \frac{-e + \frac{2 J \left( \frac{d}{d\tau} \phi(\tau) \right)}{r(\tau)}}{-1 + \frac{2 M}{r(\tau)}}$$

```
> eq53p := subs(eq53, eq34);
```

$$eq53p := \left( r(\tau)^2 + \frac{J^2}{M^2} + \frac{2J^2}{Mr(\tau)} \right) \left( \frac{d}{d\tau} \phi(\tau) \right) - \frac{2J \left( -e + \frac{2J \left( \frac{d}{d\tau} \phi(\tau) \right)}{r(\tau)} \right)}{r(\tau) \left( -1 + \frac{2M}{r(\tau)} \right)} = l$$

> eq54 := isolate(eq53p, diff(phi(tau),tau));

$$eq54 := \frac{d}{d\tau} \phi(\tau) = \frac{l M^2 (-r(\tau) + 2M) - 2J M^2 e}{-r(\tau)^3 M^2 + 2r(\tau)^2 M^3 - J^2 r(\tau)}$$

> eq54p := subs(eq54, eq53);

$$eq54p := \frac{d}{d\tau} t(\tau) = \frac{-e + \frac{2J(l M^2 (-r(\tau) + 2M) - 2J M^2 e)}{r(\tau) (-r(\tau)^3 M^2 + 2r(\tau)^2 M^3 - J^2 r(\tau))}}{-1 + \frac{2M}{r(\tau)}}$$

> eq55 := subs({eq53, eq54}, eq26);

$$eq55 := \frac{r(\tau) \left( \frac{d}{d\tau} r(\tau) \right)^2}{r(\tau)^2 - 2Mr(\tau) + \frac{J^2}{M^2}} - \frac{r(\tau)^2 \left( \frac{d}{d\tau} r(\tau) \right) \left( 2r(\tau) \left( \frac{d}{d\tau} r(\tau) \right) - 2M \left( \frac{d}{d\tau} r(\tau) \right) \right)}{\left( r(\tau)^2 - 2Mr(\tau) + \frac{J^2}{M^2} \right)^2}$$

$$+ \frac{r(\tau)^2 \left( \frac{d^2}{d\tau^2} r(\tau) \right)}{r(\tau)^2 - 2Mr(\tau) + \frac{J^2}{M^2}} + \frac{M \left( -e + \frac{2J \left( \frac{d}{d\tau} \phi(\tau) \right)}{r(\tau)} \right)^2}{r(\tau)^2 \left( -1 + \frac{2M}{r(\tau)} \right)^2}$$

$$+ \frac{1}{2} \frac{r(\tau)^2 \left( \frac{d}{d\tau} r(\tau) \right)^2 (2r(\tau) - 2M)}{\left( r(\tau)^2 - 2Mr(\tau) + \frac{J^2}{M^2} \right)^2}$$

$$- \frac{1}{2} \frac{\left( 2r(\tau) - \frac{2J^2}{Mr(\tau)^2} \right) (l M^2 (-r(\tau) + 2M) - 2J M^2 e)^2}{(-r(\tau)^3 M^2 + 2r(\tau)^2 M^3 - J^2 r(\tau))^2}$$

$$- \frac{2J \left( -e + \frac{2J \left( \frac{d}{d\tau} \phi(\tau) \right)}{r(\tau)} \right) (l M^2 (-r(\tau) + 2M) - 2J M^2 e)}{r(\tau)^2 \left( -1 + \frac{2M}{r(\tau)} \right) (-r(\tau)^3 M^2 + 2r(\tau)^2 M^3 - J^2 r(\tau))} = 0$$

Lösung der DGL mit den angenommenen init-Bedingungen

```
> M := 1; J := 0.37;
```

```
M := 1  
J := 0.37
```

```
> l := 4.7996; e := 0.9772;
```

```
l := 4.7996  
e := 0.9772
```

```
> eq77 := r(0) = 26;
```

```
eq77 := r(0) = 26
```

```
> eq78 := D(r)(0) = 0;
```

```
eq78 := D(r)(0) = 0
```

```
> eq79 := phi(0) = 0;
```

```
eq79 := phi(0) = 0
```

```
> eq80 := D(phi)(0) = eval(rhs(eq54), {r(tau)=rhs(eq77)});
```

```
eq80 := D(phi)(0) = 0.007143004388
```

```
> ini := eq77, eq78, eq79;
```

```
ini := r(0) = 26, D(r)(0) = 0, phi(0) = 0
```

```
> eq91 := dsolve({eq54, eq55, ini}, {r(tau), phi(tau)},  
numeric,
```

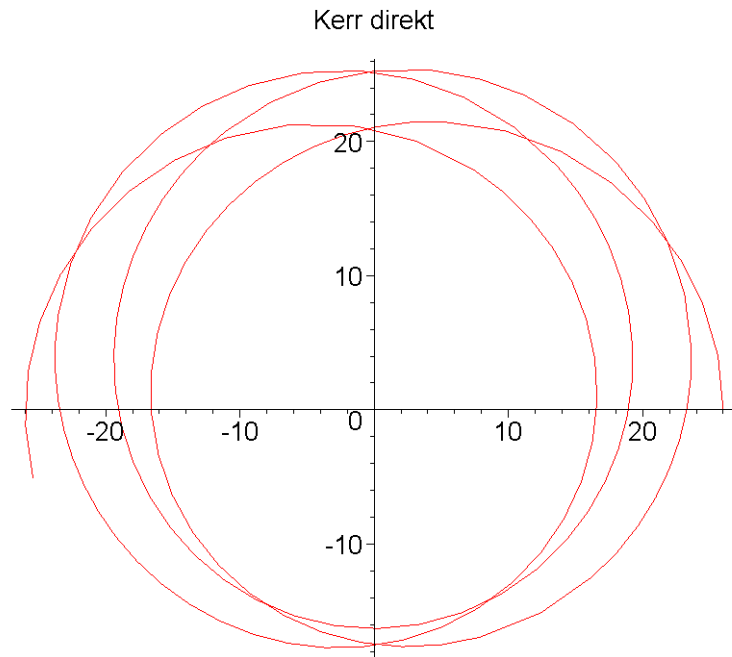
```
output=listprocedure);
```

```
eq91 := [tau = (proc(tau) ... end proc), phi(tau) = (proc(tau) ... end proc),
```

```
r(tau) = (proc(tau) ... end proc),  $\frac{d}{d\tau} r(tau) = (proc(tau) ... end proc)$ ]
```

```
> polarplot([rhs(eq91(tau)[3]), rhs(eq91(tau)[2]),  
tau=0..2000],
```

```
scaling=constrained, color=red, title="Kerr  
direkt", legend="Umlaufbahn rechts auf x-Achse beginnend");
```



Umlaufbahn rechts auf x-Achse beginnend

```

> pkerrd := polarplot([rhs(eq91(tau)[3]), rhs(eq91(tau)[2]),
tau=0..1000],
scaling=constrained, color=red, legend="Kerr
direct"):

> M := 1; J := -0.37;
M:= 1
J:= -0.37

> l := 4.7996; e := 0.976797;
l:= 4.7996
e:= 0.976797

> eq77 := r(0) = 26;
eq77 := r(0) = 26

> eq78 := D(r)(0) = 0;
eq78 := D(r)(0) = 0

> eq79 := phi(0) = 0;
eq79 := phi(0) = 0

> eq80 := D(phi)(0) = eval(rhs(eq54), {r(tau)=rhs(eq77)});
eq80 := D(phi)(0) = 0.007053899319

> ini := eq77, eq78, eq79;
ini := r(0) = 26, D(r)(0) = 0, phi(0) = 0

> eq91 := dsolve({eq54, eq55, ini}, {r(tau), phi(tau)},
numeric,
output=listprocedure);

```

```

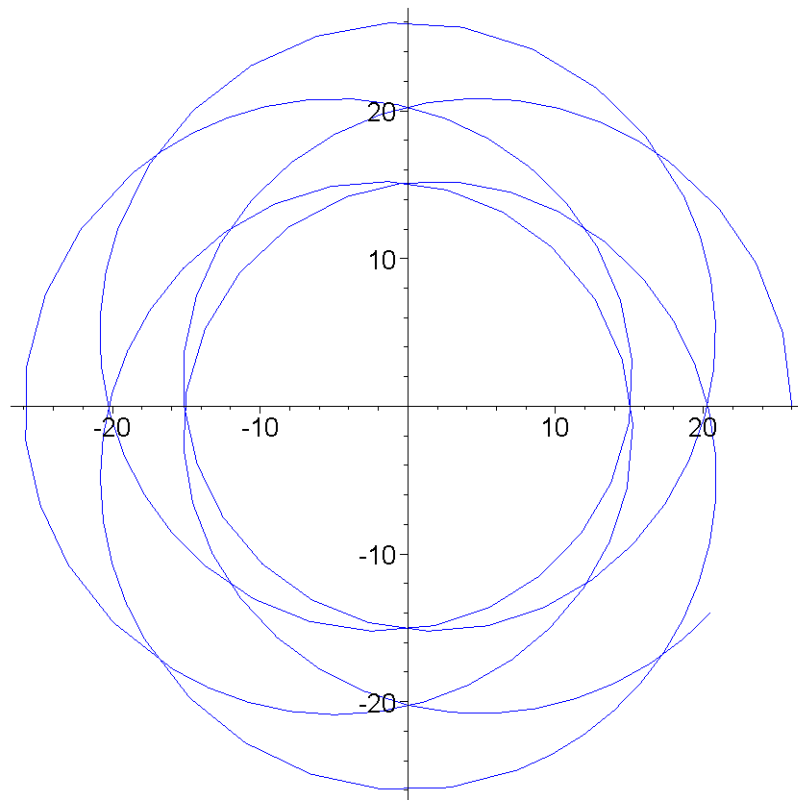
eq91 := [ tau = (proc(tau) ... end proc), phi(tau) = (proc(tau) ... end proc),
          r(tau) = (proc(tau) ... end proc), d/dtau r(tau) = (proc(tau) ... end proc) ]

```

```

> polarplot([rhs(eq91(tau)[3]), rhs(eq91(tau)[2]),
tau=0..2500],
            scaling=constrained,color=blue, title="Kerr
retro",legend="Umlaufbahn rechts auf x-Achse beginnend");

```



Umlaufbahn rechts auf x-Achse beginnend

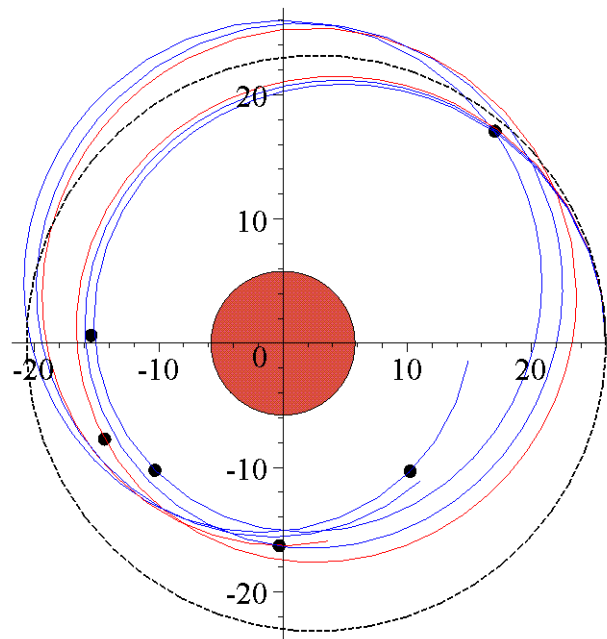
```

> pkerro := polarplot([rhs(eq91(tau)[3]), rhs(eq91(tau)[2]),
tau=0..1000],
                    scaling=constrained, color=blue, legend="Kerr
retro", axesfont=[TIMES, ROMAN, 12]):
> pkerr1 := disk([16.31*cos(3.63467), 16.31*sin(3.63467)], 0.5,
color=black):
> pkerr2 := disk([14.53*cos(3.923), 14.53*sin(3.923)], 0.5,
color=black):
> pkerr3 := disk([16.31*cos(10.9765), 16.31*sin(10.9765)], 0.5,
color=black):
> pkerr4 := disk([14.53*cos(11.7778), 14.53*sin(11.7778)], 0.5,

```

```
color=black):
```

```
> display([pnewton,pschwarz,pkerrd,pkerro,pkerr1,pkerr2,psch2,p  
kerr3,pkerr4,psch3,pns]);
```



----- Newton Lösung  
----- Schwarzschild  
----- Kerr direct  
----- Kerr retro

## - Das "Gummibett"

Die Gummibettgleichung, siehe: Misner, Thorne, and Wheeler, *Gravitation*, p. 613. Der Stern wird mit gleichförmig verteilter Masse angenommen

```
> M := 'M': m := rp^3/R^3*M;
```

$$m := \frac{rp^3 M}{R^3}$$

```
> zin := int(sqrt(1/(1 - 2*m/rp) - 1), rp=2*M..r) + C;
```

$$zin := \int_{2M}^r \sqrt{\frac{1}{1 - \frac{2rp^2 M}{R^3}} - 1} dr + C$$

```
> zout := int(sqrt(1/(1 - 2*M/rp) - 1), rp=2*M..r);
```



$$z_{out} := 2\sqrt{2} \sqrt{\frac{M}{r-2M}} (r-2M)$$

```

> eq201 := eval(zin, r=R) = eval(zout, r=R):
> C := solve(eq201, C):
> pin := plot3d([r*cos(phi), r*sin(phi), eval(zin, {M=1,
R=5.8})], phi=0..2*Pi, r=0..5.8, grid=[25,6],
orientation=[88,135], shading=zhue):
> pout := plot3d([r*cos(phi), r*sin(phi), eval(zout, {M=1,
R=5.8})], phi=0..2*Pi, r=5.8..27, grid=[25,12], axes=boxed):

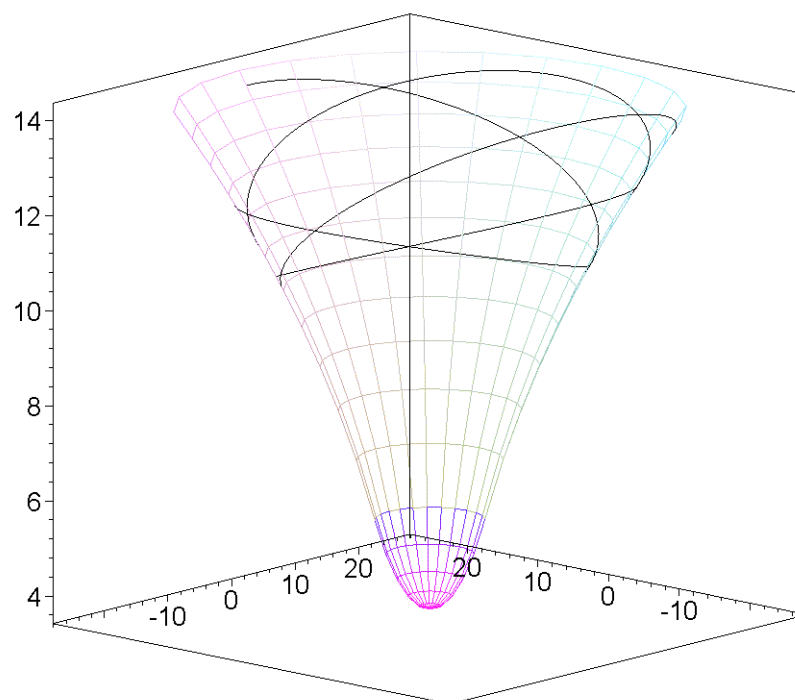
```

Die Trajektorie (Bahnkurve) wird in diese Gummibett Darstellung eingemalt. Diesen Federball kann man drehen und wenden und sich die Kurven betrachten.

```

> display([pin, pout, pschwarz3d], style=hidden);

```



## 1. Beispiel

ein Partikel mit einem 0 Grad Moment wird von einem Schwarzen Loch "Kerr Typ"

eingefangen; mit max. Spin-Winkel-Moment  $J$  ( $a = M$ ) angenommen.

> restart:

> a := J/M; Delta := r(tau)^2 - 2\*M\*r(tau) + a^2;

$$a := \frac{J}{M}$$

$$\Delta := r(\tau)^2 - 2 M r(\tau) + \frac{J^2}{M^2}$$

> f := 1/2\*(-(1 - 2\*M/r(tau))\*diff(t(tau),tau)^2 + r(tau)^2/(Delta)\*diff(r(tau),tau)^2 + (r(tau)^2 + a^2 + 2\*M\*a^2/r(tau))\*diff(phi(tau),tau)^2 - 4\*a\*M/r(tau)\*diff(t(tau),tau)\*diff(phi(tau),tau));

$$f := -\frac{1}{2} \left( 1 - \frac{2M}{r(\tau)} \right) \left( \frac{d}{d\tau} t(\tau) \right)^2 + \frac{1}{2} \frac{r(\tau)^2 \left( \frac{d}{d\tau} r(\tau) \right)^2}{r(\tau)^2 - 2 M r(\tau) + \frac{J^2}{M^2}} + \frac{1}{2} \left( r(\tau)^2 + \frac{J^2}{M^2} + \frac{2 J^2}{M r(\tau)} \right) \left( \frac{d}{d\tau} \phi(\tau) \right)^2 - \frac{2 J \left( \frac{d}{d\tau} t(\tau) \right) \left( \frac{d}{d\tau} \phi(\tau) \right)}{r(\tau)}$$

> f1 := subs({t(tau)=var1, diff(t(tau),tau)=var2, r(tau)=var3, diff(r(tau),tau)=var4, phi(tau)=var5, diff(phi(tau),tau)=var6}, f):

> e11 := diff(f1, var2):

> e12 := diff(f1, var1):

> e13 := subs({var1=t(tau), var2=diff(t(tau),tau), var3=r(tau), var5=phi(tau), var4=diff(r(tau),tau), var6=diff(phi(tau),tau)}, e11):

> eq14 := e13 = -e;

$$eq14 := -\left( 1 - \frac{2M}{r(\tau)} \right) \left( \frac{d}{d\tau} t(\tau) \right) - \frac{2 J \left( \frac{d}{d\tau} \phi(\tau) \right)}{r(\tau)} = -e$$

> e21 := diff(f1, var4):

> e22 := diff(f1, var3):

> e23 := subs({var1=t(tau), var2=diff(t(tau),tau), var3=r(tau), var5=phi(tau), var4=diff(r(tau),tau), var6=diff(phi(tau),tau)}, e21):

> e24 := subs({var1=t(tau), var2=diff(t(tau),tau), var3=r(tau), var5=phi(tau), var4=diff(r(tau),tau), var6=diff(phi(tau),tau)}, e22):

[ > e25 := diff(e23, tau):

[ > eq26 := e25 - e24 = 0;

$$eq26 := \frac{r(\tau) \left( \frac{d}{d\tau} r(\tau) \right)^2}{r(\tau)^2 - 2 M r(\tau) + \frac{J^2}{M^2}} - \frac{r(\tau)^2 \left( \frac{d}{d\tau} r(\tau) \right) \left( 2 r(\tau) \left( \frac{d}{d\tau} r(\tau) \right) - 2 M \left( \frac{d}{d\tau} r(\tau) \right) \right)}{\left( r(\tau)^2 - 2 M r(\tau) + \frac{J^2}{M^2} \right)^2}$$

$$+ \frac{r(\tau)^2 \left( \frac{d^2}{d\tau^2} r(\tau) \right)}{r(\tau)^2 - 2 M r(\tau) + \frac{J^2}{M^2}} + \frac{M \left( \frac{d}{d\tau} t(\tau) \right)^2}{r(\tau)^2} + \frac{1}{2} \frac{r(\tau)^2 \left( \frac{d}{d\tau} r(\tau) \right)^2 (2 r(\tau) - 2 M)}{\left( r(\tau)^2 - 2 M r(\tau) + \frac{J^2}{M^2} \right)^2}$$

$$- \frac{1}{2} \left( 2 r(\tau) - \frac{2 J^2}{M r(\tau)^2} \right) \left( \frac{d}{d\tau} \phi(\tau) \right)^2 - \frac{2 J \left( \frac{d}{d\tau} t(\tau) \right) \left( \frac{d}{d\tau} \phi(\tau) \right)}{r(\tau)^2} = 0$$

[ > e31 := diff(f1, var6):

[ > e32 := diff(f1, var5):

[ > e33 := subs({var1=t(tau), var2=diff(t(tau),tau), var3=r(tau), var5=phi(tau), var4=diff(r(tau),tau), var6=diff(phi(tau),tau)}, e31):

[ > eq34 := e33 = l;

$$eq34 := \left( r(\tau)^2 + \frac{J^2}{M^2} + \frac{2 J^2}{M r(\tau)} \right) \left( \frac{d}{d\tau} \phi(\tau) \right) - \frac{2 J \left( \frac{d}{d\tau} t(\tau) \right)}{r(\tau)} = l$$

[ > eq53 := isolate(eq14, diff(t(tau),tau));

$$eq53 := \frac{d}{d\tau} t(\tau) = \frac{-e + \frac{2 J \left( \frac{d}{d\tau} \phi(\tau) \right)}{r(\tau)}}{-1 + \frac{2 M}{r(\tau)}}$$

[ > eq53p := subs(eq53, eq34);

$$eq53p := \left( r(\tau)^2 + \frac{J^2}{M^2} + \frac{2 J^2}{M r(\tau)} \right) \left( \frac{d}{d\tau} \phi(\tau) \right) - \frac{2 J \left( -e + \frac{2 J \left( \frac{d}{d\tau} \phi(\tau) \right)}{r(\tau)} \right)}{r(\tau) \left( -1 + \frac{2 M}{r(\tau)} \right)} = l$$

[ > eq54 := isolate(eq53p, diff(phi(tau),tau));

$$eq54 := \frac{d}{d\tau} \phi(\tau) = \frac{-l M^2 (-r(\tau) + 2 M) + 2 J M^2 e}{r(\tau)^3 M^2 - 2 r(\tau)^2 M^3 + J^2 r(\tau)}$$

[ > eq54p := subs(eq54, eq53);

$$eq54p := \frac{d}{d\tau} t(\tau) = \frac{-e + \frac{2J(-lM^2(-r(\tau) + 2M) + 2JM^2e)}{r(\tau)(r(\tau)^3M^2 - 2r(\tau)^2M^3 + J^2r(\tau))}}{-1 + \frac{2M}{r(\tau)}}$$

> **eq55 := subs({eq53, eq54}, eq26);**

$$eq55 := \frac{r(\tau) \left( \frac{d}{d\tau} r(\tau) \right)^2}{r(\tau)^2 - 2Mr(\tau) + \frac{J^2}{M^2}} - \frac{r(\tau)^2 \left( \frac{d}{d\tau} r(\tau) \right) \left( 2r(\tau) \left( \frac{d}{d\tau} r(\tau) \right) - 2M \left( \frac{d}{d\tau} r(\tau) \right) \right)}{\left( r(\tau)^2 - 2Mr(\tau) + \frac{J^2}{M^2} \right)^2}$$

$$+ \frac{r(\tau)^2 \left( \frac{d^2}{d\tau^2} r(\tau) \right)}{r(\tau)^2 - 2Mr(\tau) + \frac{J^2}{M^2}} + \frac{M \left( -e + \frac{2J \left( \frac{d}{d\tau} \phi(\tau) \right)}{r(\tau)} \right)^2}{r(\tau)^2 \left( -1 + \frac{2M}{r(\tau)} \right)^2}$$

$$+ \frac{1}{2} \frac{r(\tau)^2 \left( \frac{d}{d\tau} r(\tau) \right)^2 (2r(\tau) - 2M)}{\left( r(\tau)^2 - 2Mr(\tau) + \frac{J^2}{M^2} \right)^2}$$

$$- \frac{1}{2} \frac{\left( 2r(\tau) - \frac{2J^2}{Mr(\tau)^2} \right) (-lM^2(-r(\tau) + 2M) + 2JM^2e)^2}{(r(\tau)^3M^2 - 2r(\tau)^2M^3 + J^2r(\tau))^2}$$

$$- \frac{2J \left( -e + \frac{2J \left( \frac{d}{d\tau} \phi(\tau) \right)}{r(\tau)} \right) (-lM^2(-r(\tau) + 2M) + 2JM^2e)}{r(\tau)^2 \left( -1 + \frac{2M}{r(\tau)} \right) (r(\tau)^3M^2 - 2r(\tau)^2M^3 + J^2r(\tau))} = 0$$

> **M := 1; J := 1;**

M := 1

J := 1

> **l := 0; e := 1;**

l := 0

e := 1

> **eq77 := r(0) = 5;**

eq77 := r(0) = 5

> **eq78 := D(r)(0) = -0.64;**

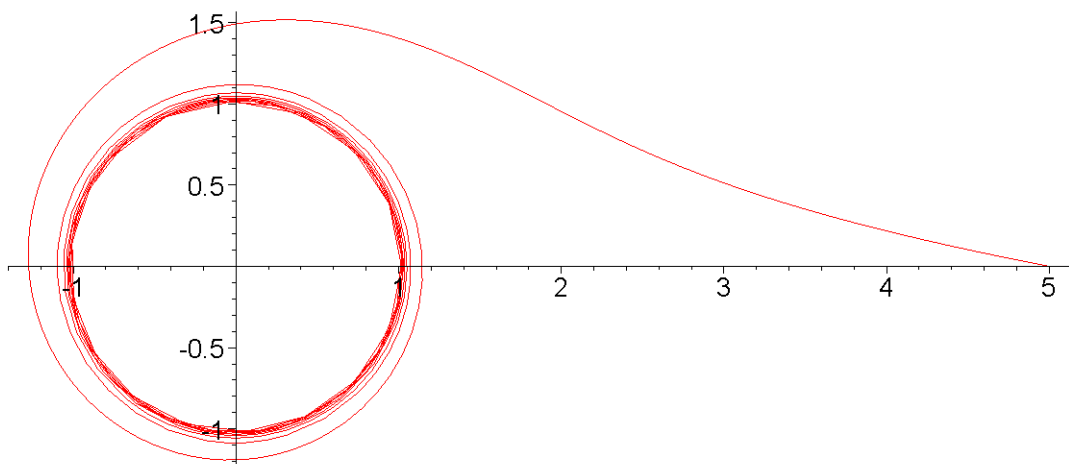
eq78 := D(r)(0) = -0.64

> **eq79 := phi(0) = 0;**

```

eq79 := phi(0) = 0
> eq80 := D(phi)(0) = eval(rhs(eq54), {r(tau)=rhs(eq77)});
eq80 := D(phi)(0) = 1/40
> ini := eq77, eq78, eq79;
ini := r(0) = 5, D(r)(0) = -0.64, phi(0) = 0
> eq91 := dsolve({eq54, eq55, ini}, {r(tau), phi(tau)},
numeric,
output=listprocedure);
eq91 := [tau = (proc(tau) ... end proc), phi(tau) = (proc(tau) ... end proc),
r(tau) = (proc(tau) ... end proc), d/dtau r(tau) = (proc(tau) ... end proc)]
> with(plots):
Warning, the name changecoords has been redefined
> polarplot([rhs(eq91(tau)[3]), rhs(eq91(tau)[2]),
tau=0..4.49],
scaling=constrained, numpoints=2400);

```



## 2. Beispiel

Hier wird das Verhalten eines Partikels nahe des instabilen Orbits demonstriert. Das Partikelchen umkreist den Körper einige male und wird dann rausgeschleudert. Klar, das werden wir auch animieren.

> restart:

> f := 1/2\*(-(c^2 - 2\*G\*M/r(s))\*diff(t(s),s)^2 + 1/(1 - 2\*G\*M/(c^2\*r(s)))\*diff(r(s),s)^2 + r(s)^2\*diff(phi(s),s)^2);

$$f := -\frac{1}{2} \left( c^2 - \frac{2GM}{r(s)} \right) \left( \frac{d}{ds} t(s) \right)^2 + \frac{1}{2} \frac{\left( \frac{d}{ds} r(s) \right)^2}{1 - \frac{2GM}{c^2 r(s)}} + \frac{1}{2} r(s)^2 \left( \frac{d}{ds} \phi(s) \right)^2$$

> f1 := subs({t(s)=var1, diff(t(s),s)=var2, r(s)=var3, diff(r(s),s)=var4, phi(s)=var5, diff(phi(s),s)=var6}, f):

> e11 := diff(f1, var2):

> e12 := diff(f1, var1):

> e13 := subs({var1=t(s), var2=diff(t(s),s), var3=r(s), var5=phi(s), var4=diff(r(s),s), var6=diff(phi(s),s)}, e11):

> eq14 := e13 = -a/c;

$$eq14 := -\left( c^2 - \frac{2GM}{r(s)} \right) \left( \frac{d}{ds} t(s) \right) = -\frac{a}{c}$$

> e21 := diff(f1, var4):

> e22 := diff(f1, var3):

> e23 := subs({var1=t(s), var2=diff(t(s),s), var3=r(s), var5=phi(s), var4=diff(r(s),s), var6=diff(phi(s),s)}, e21):

> e24 := subs({var1=t(s), var2=diff(t(s),s), var3=r(s), var5=phi(s), var4=diff(r(s),s), var6=diff(phi(s),s)}, e22):

> e25 := diff(e23, s):

> eq26 := e25 - e24 = 0;

$$eq26 := -\frac{\left( \frac{d}{ds} r(s) \right)^2 GM}{\left( 1 - \frac{2GM}{c^2 r(s)} \right)^2 c^2 r(s)^2} + \frac{\frac{d^2}{ds^2} r(s)}{1 - \frac{2GM}{c^2 r(s)}} + \frac{GM \left( \frac{d}{ds} t(s) \right)^2}{r(s)^2} - r(s) \left( \frac{d}{ds} \phi(s) \right)^2 = 0$$

> e31 := diff(f1, var6):

> e32 := diff(f1, var5):

> e33 := subs({var1=t(s), var2=diff(t(s),s), var3=r(s), var5=phi(s), var4=diff(r(s),s), var6=diff(phi(s),s)}, e31):

> eq34 := e33 = h/c;

$$eq34 := r(s)^2 \left( \frac{d}{ds} \phi(s) \right) = \frac{h}{c}$$

```

> eq53 := isolate(eq14, diff(t(s),s));

$$eq53 := \frac{d}{ds} t(s) = -\frac{a}{c \left( -c^2 + \frac{2GM}{r(s)} \right)}$$

> eq54 := isolate(eq34, diff(phi(s),s));

$$eq54 := \frac{d}{ds} \phi(s) = \frac{h}{c r(s)^2}$$

> eq55 := eval(eq26, {eq53, eq54});

$$eq55 := -\frac{\left( \frac{d}{ds} r(s) \right)^2 GM}{\left( 1 - \frac{2GM}{c^2 r(s)} \right)^2 c^2 r(s)^2} + \frac{\frac{d^2}{ds^2} r(s)}{1 - \frac{2GM}{c^2 r(s)}} + \frac{GM a^2}{r(s)^2 c^2 \left( -c^2 + \frac{2GM}{r(s)} \right)^2} - \frac{h^2}{r(s)^3 c^2} = 0$$

> M := 1; G := 1; c := 1;

$$M := 1$$


$$G := 1$$


$$c := 1$$

> eq77 := r(0) = 3.76777;

$$eq77 := r(0) = 3.76777$$

> eq78 := D(r)(0) = 0;

$$eq78 := D(r)(0) = 0$$

> eq79 := phi(0) = 0;

$$eq79 := \phi(0) = 0$$

> eq80 := D(phi)(0) = 0.30290054;

$$eq80 := D(\phi)(0) = 0.30290054$$

> h := eval(lhs(eq34), {r(s)=rhs(eq77),
diff(phi(s),s)=rhs(eq80)});

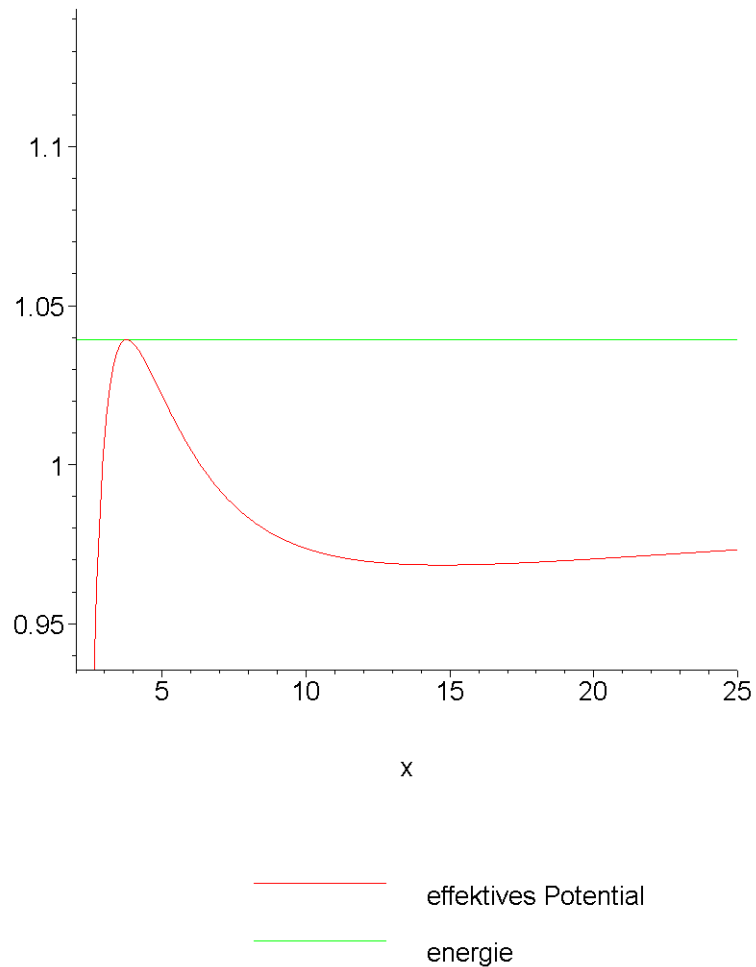
$$h := 4.300003560$$

> a := sqrt((1 - 2*M/rhs(eq77))*(1 + h^2/rhs(eq77)^2));

$$a := 1.039364786$$

> Vsqr := sqrt((1 - 2*M/x)*(1 + h^2/x^2)):
> plot([Vsqr, a], x=2..25, a-0.1*a..a+0.1*a, legend=["effektives
Potential", "energie"]);

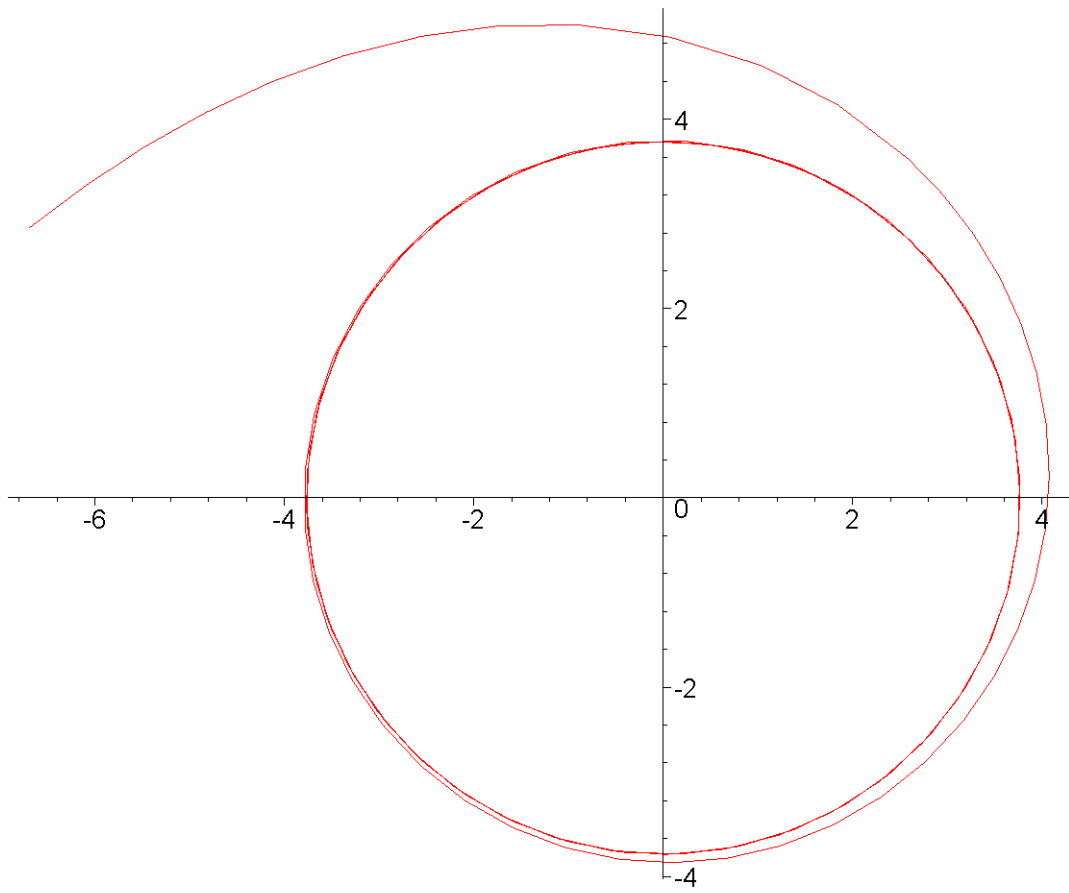
```



```

> ini := eq77, eq78, eq79;
      ini := r(0) = 3.76777, D(r)(0) = 0, phi(0) = 0
> eq91 := dsolve({eq54, eq55, ini}, {r(s), phi(s)}, numeric,
  output=listprocedure);
eq91 := [ s = (proc(s) ... end proc), phi(s) = (proc(s) ... end proc),
  r(s) = (proc(s) ... end proc),  $\frac{d}{ds} r(s) = (\text{proc}(s) \dots \text{end proc}) ]$ 
> with(plots):
Warning, the name changecoords has been redefined
> polarplot([rhs(eq91(s)[3]), rhs(eq91(s)[2]), s=0..100],
  scaling=constrained);
  
```

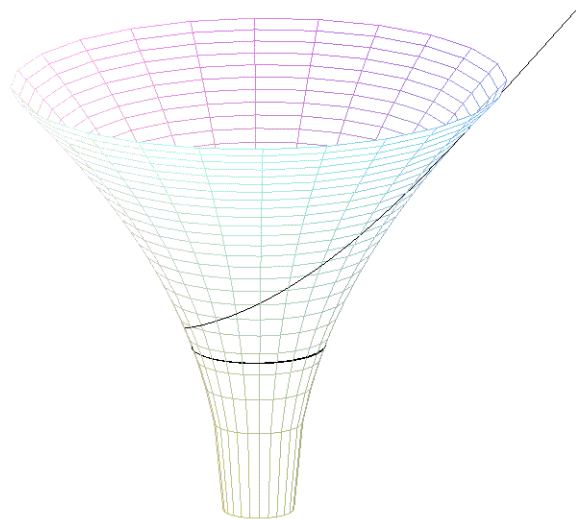




```

> p1 := plot3d([x*cos(t), x*sin(t), sqrt(8*(x-2))], x=2..14,
  t=0..2*Pi, style=hidden):
> p2 := spacecurve([rhs(eq91(s)[3]), rhs(eq91(s)[2]),
  sqrt(8*(rhs(eq91(s)[3])-2)), s=0..130], coords=cylindrical,
  color=black, numpoints=400):
> display([p1, p2]);

```



```
> N := 75; nstep := 2;
```

```
N := 75
```

```
nstep := 2
```

```
> for i from 0 by nstep to N*nstep do
```

```
    pl[i] := polarplot([rhs(eq91(s)[3]), rhs(eq91(s)[2]),
s=i..i+nstep], numpoints=5):
```

```
    pl3d[i] := spacecurve([rhs(eq91(s)[3]), rhs(eq91(s)[2]),
sqrt(8*(rhs(eq91(s)[3])-2)), s=i..i+nstep],
coords=cylindrical, numpoints=5, color=black):
```

```
    plsum[i] := display(seq(pl[j*nstep], j=0..i/nstep)):
```

```
    pl3dsum[i] := display({pl, seq(pl3d[j*nstep],
j=0..i/nstep)}):
```

```
end do:
```

```
> display([seq(plsum[i*nstep], i=0..N)], insequence=true,
scaling=constrained);
```

